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# On invariant probability measure of a piecewise-smooth circle homeomorphism of Zygmund class

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## ABSTRACT

We show that the invariant probability measure of the ergodic piecewise-smooth circle homeomorphisms with several break points which satisfy Zygmund condition and the product of jumps at break points non-trivial is singular with respect to Lebesgue measure.

**Keywords:** break point, circle homeomorphism, invariant measure, rotation number

# 1. Introduction

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle with clearly defined orientation, metric, Lebesgue measure and the operation of addition. Let $\pi : \mathbb{R} \to S^1$ denote the corresponding projection mapping that "winds" a straight line on the circle. An arbitrary homeomorphism $f$ that preserves the orientation of the unit circle $S^1$ can be "lifted" on the straight line $\mathbb{R}$ in the form of the homeomorphism $F : \mathbb{R} \to \mathbb{R}$ with property $F(x+1) = F(x) + 1$ that is connected with $f$ by relation $\pi \circ F = f \circ \pi$. This homeomorphism $F$ is called the *lift* of the homeomorphism $f$ and is defined up to an integer term. The most important arithmetic characteristic of the homeomorphism $f$ of the unit circle $S^1$ is the *rotation number*

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \ mod\ 1$$

where $F$ is the lift of $f$ with $S^1$ to $\mathbb{R}$. Here and below, for a given map $F$, $F^n$ denotes its $n$-th iterate. The rotation number is rational if and only if $f$ has periodic points. Poincare proved that, if $f$ does not have any periodic orbit then it is semi-conjugate to the linear rotation $f_\rho : x \to x + \rho\ mod\ 1$. Denjoy (1932) proved that,if $f$ is a circle diffeomorphism with irrational rotation number $\rho = \rho(f)$ and $\log Df$ is of bounded variation, then $f$ is *topologically conjugate* to the pure rotation $f_\rho : x \to x + \rho\ mod\ 1$, that is, there exists an essentially unique homeomorphism $\varphi$ of the circle with $\varphi \circ f = f_\rho \circ \varphi$. It is well known that every circle homeomorphism with irrational rotation number $\rho$ has a unique $f-$invariant probability measure $\mu_f$. Furthermore, the conjugation $\varphi$ and the invariant probability measure $\mu_f$ are connected by the relation $\varphi(x) = \mu_f([0, x])$, $x \in S^1$ (see, for example, Cornfeld et al. (1982)). Because of this relation, $\mu_f$ is absolutely continuous with respect to the Lebesgue measure $l$ if and only if $\varphi$ is given by an absolutely continuous function. The invariant probability measures of circle homeomorphisms with one break point was studied for the first time by Dzhalilov and Khanin (1998). Later Dzhalilov et al. (2012) extended this result for the case of circle homeomorphisms with several break points. In all above cases the invariant probability measures of circle homeomorphisms with break points are singular with respect to Lebesgue measure. The result of Dzhalilov et al. (2012) on circle homeomorphisms with $n$ break points is the following theorem.

**Theorem 1.1.** *Dzhalilov et al. (2012). Let $f : S^1 \to S^1$ be an orientation preserving circle homeomorphism with irrational rotation number $\rho$, and satisfies the following conditions:*

**(a)** *There exist points $b(1), b(2), ..., b(n) \in S^1$ at which the derivatives $f'(b(i)\tfrac{1}{2}0) > 0$ are defined and $f'(b(i) - 0)/f'(b(i) + 0) = \sigma_i(b(i), f) \neq 1, i = 1, ..., n$;*

**(b)** *$f'(x)$ is an absolutely continuous function on every connected component of the set $S^1 \setminus \{b(i), i = 1, ..., n\}$;*

**(c)** *$f''(x) \in L_1(S^1, dl)$;*

**(d)** $\prod_{i=1}^{n} \sigma_i(b(i), f) \neq 1.$

*Then the invariant probability measure $\mu_f$ is singular with respect to Lebesgue measure $l$ on the circle $S^1$.*

Where the points $b(1), b(2), ..., b(n)$ are called break points and the numbers $\sigma_i(b(i), f)$ are jumps at break points of $f$.

Before formulate our main result we introduce the following class of homeomorphisms. Let $\psi : S^1 \to S^1$ be a continuous, non decreasing function with $\psi(0) = 0$. Using this function we define a class of orientation preserving circle homeomorphisms $f$ such that

$$|f(x+t) + f(x-t) - 2f(x)| \leq Ct\psi(t) \tag{1}$$

for all $x, t \in S^1$ such that $x - t, x + t \in S^1$, here $C > 0$ is a constant. The class of real functions satisfying (1) with $\psi \equiv 1$ on real line is called Zygmund class and denoted by $\Lambda_*$ (see Zygmund (2002)). This class plays a key role to investigate the trigonometric series. The class $\Lambda_*$ was applied to the circle homeomorphisms for the first time by Hu and Sullivan (1997). They extended the classical Denjoy's theorem to this class. The functions satisfying (1) are not of bounded variation at all, the reverse also is not true. For example let us consider Weieratrass function:

$$W_\beta(x) = \sum_{n=1}^{\infty} \theta_n b^{-n\beta} \cos(b^n x)$$

where $b > 1$ and $\lim_{n\to\infty} \theta_n = 0$. The following fact can be found in Zygmund (2002). Weieratrass proved that for a small enough $\beta > 0$ the function $W_\beta$ is nowhere differentiable. The extension to $\beta \leq 1$ was fist proved by Hardy. For $\beta > 1$ the function $W'_\beta$ exists and continuous. If the sum of squares of the sequence $\theta_n$ is divergence then $W_1$ is differentiable in a set of measure zero. Thus making $b$ even number and instead of $\theta_n$ taking the sequence $n^{-1/2}$ easily we may check that the function $W_1$ satisfies the condition (1) but almost nowhere differentiable.

The main result of this paper is the following theorem.

**Theorem 1.2.** *Let $f$ be a $C^1$ circle diffeomorphism with irrational rotation number $\rho$, and satisfies the following conditions:*

**(1)** *There exist points $b(1), b(2), ..., b(n) \in S^1$ at which the derivatives $f'(b(i)\frac{1}{2}0) > 0$ are defined and $f'(b(i)-0)/f'(b(i)+0) = \sigma_i(b(i), f) \neq 1, i = 1, ..., n$;*

**(2)** $f'(x) \in C^Z(S^1 \setminus \{b(i), i = 1, ..., n\})$, *$f'(x) > 0$ for all $x \in S^1 \setminus \{b(i), i = 1, ..., n\}$;*

**(3)** $\prod_{i=1}^{n} \sigma_i(b(i), f) \neq 1$.

*Then the invariant probability measure $\mu_f$ is singular with respect to Lebesgue measure $l$ on the circle $S^1$.*

## 2. Necessary facts and definitions

An ordering on the circle $S^1$ is determined by an orientation of it. Let $a, b, c$ be points on $S^1$. We write $a \prec b \prec c$ to signify that when moving from $a$ in the positive direction, we first reach $b$, and then $c$. We consider a circle homeomorphism $f$ that preserves orientation and has irrational rotation number $\rho$. Let $\{a_k, k \in \mathbb{N}\}$ denote the sequence of elements in the expansion of $\rho$ into a continued fraction, that is, $\rho = [a_1, a_2, ..., a_n, ...]$. We set $p_n/q_n = [a_1, a_2, ..., a_n], n \geq 1$. The numbers $p_n/q_n$ are called the *convergents* of $\rho$, and $q_n$ is the *first return time*. The numbers $q_n$ satisfy the difference equation $q_{n+1} = a_{n+1}q_n + q_{n-1}, n \geq 1$, with the initial conditions $q_0 = 1$ and $q_1 = a_1$. For an arbitrary point $x_0 \in S^1$, let $\triangle_0^{(n)}(x_0)$ denote the closed interval with endpoints $x_0$ and $x_{q_n} = f^{q_n}(x_0)$. Note that for odd $n$ the point $x_{q_n}$ lies to the left of $x_0$, and for even $n$ to the right. We set $\triangle_i^{(n)} = f^i(\triangle_0^{(n)}), i \geq 1$.

**Lemma 2.1.** *Sinai (1995) Consider an arbitrary point $x_0 \in S^1$. The segment $\{x_i, 0 \leq i < q_n + q_{n-1}\}$ of the trajectory of this point divides the circle into the following disjoint (except for the endpoints ) intervals: $\triangle_i^{(n)}$, $0 \leq i \leq q_{n-1} - 1$, $\triangle_i^{(n-1)}$, $0 \leq i \leq q_n - 1$.*

We denote the resulting partition by $\xi_n(x_0)$ and call it a *dynamical partition of order $n$.* We now describe the process of transition from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$. All the intervals $\triangle_j^{(n)}$, $0 \leq j \leq q_{n-1} - 1$ are preserved, and each of the intervals

$\triangle_i^{(n-1)},\ 0 \leq i \leq q_n - 1$ is divided into $a_{n+1}+1$ parts:

$$\triangle_i^{(n-1)}(x_0) = \triangle_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{a_{n+1}-1} \triangle_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

**Lemma 2.2.** *Consider a circle homeomorphism $f$ with lift $F$ and irrational rotation number. Suppose that at points $b(i) \in S^1, i = 1, ..., n,\ b(1) \prec b(2) \prec ... \prec b(n)$ there exist finite one-sided derivatives $F'(b(i)\frac{1}{2}0) > 0,\ F \in C^1([b(i), b(i+1)]), i = 1, ..., k,\ b(n+1) = b(1)$ and $\sum_{i=1}^{n} var_{[b(i),b(i+1)]} \ln F' = \overline{v} < \infty$. Let*

$$v = \overline{v} + \sum_{i=1}^{n} |\ln F'(b(i-0)) - \ln F'(b(i+0))|.$$

*Then the inequalities*

$$e^{-v} \leq \prod_{s=0}^{q_n-1} F'(y_s) \leq e^{v}$$

*hold for any $y_0$ such that $f^s(y_0) \in S^1 \setminus \{b(i), i = 1, 2, ..., n\},\ 0 \leq s < q_n$.*

These inequalities are called Denjoys inequalities. Lemma 2.2 is proved in the same fashion as the analogous assertion for diffeomorphisms (see Khanin and Sinai (1989)). It follows from Lemma 2.2 that the intervals comprising the dynamical partition $\xi_n(x_0)$ have exponentially small lengths. We set $\lambda = (1 + e^{-v})^{-1/2} < 1$.

**Corollary 2.1.** *Let $\triangle^{(n)}$ be an arbitrary element of the dynamical partition $\xi_n(x_0)$. Then*

$$l(\triangle^{(n)}) \leq C_0 \lambda^n,$$

*where the constant $C_0$ is independent of $n$ and $x_0$.*

**Denjoys Theorem** Khanin and Sinai (1989). *Suppose that the hypotheses of Lemma 2.2 hold. Then the homeomorphism $f$ is topologically conjugate to the linear rotation $f_\rho$.*

**Definition 2.1.** *Let $K > 1$ be a constant. Two intervals $I_1$ and $I_2$ are said to be $K$-commensurable on $S^1$ if the inequalities $K^{-1}l(I_2) \leq l(I_1) \leq Kl(I_2)$ hold.*

In accordance with Khanin and Sinai (1989) we introduce the following definition.

**Definition 2.2.** *An interval $I = [\tau, t] \subset S^1$ is said to be $q_n$-small, and its endpoints $q_n$-close, if the intervals $f^i(I)$, $0 \leq i \leq q_n - 1$, are pairwise disjoint (except for endpoints).*

It follows from the structure of dynamical partitions that an interval $I = [\tau, t]$ is $q_n$-small if and only if either $\tau < t \leq f^{q_n-1}(\tau)$ or $f^{q_n-1}(t) \leq \tau < t$.

**Definition 2.3.** *The cross-ratio of four numbers $(z_1, z_2, z_3, z_4)$, $z_1 < z_2 < z_3 < z_4$, is the number*

$$Cr(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

The notion of cross-ratios plays an important role in what follows.

**Definition 2.4.** *Suppose that we are given four numbers $(z_1, z_2, z_3, z_4)$, $z_1 < z_2 < z_3 < z_4$ and a strictly increasing function $F : \mathbb{R}^1 \to \mathbb{R}^1$. The distortion of a cross-ratio is given by*

$$Dist(z_1, z_2, z_3, z_4; F) = \frac{Cr(F(z_1), F(z_2), F(z_3), F(z_4))}{Cr(z_1, z_2, z_3, z_4)}.$$

Let $m \geq 3$, $z_i \in S_1$, $i = 1, 2, ..., m$ and suppose that $z_1 \prec z_2 \prec ... \prec z_m \prec z_1$ (in the sense of ordering on the circle). We set $\widehat{z}_1 = z_1$ and $\widehat{z}_i = \begin{cases} z_i, & if \;\; z_1 < z_i < 1 \\ 1 + z_1, & if \;\; 0 < z_1 < z_i. \end{cases}$ for $i = 2, 3, ..., m$. Obviously $\widehat{z}_1 < \widehat{z}_2 < ... < \widehat{z}_m$. The vector $(\widehat{z}_1, \widehat{z}_2, ..., \widehat{z}_m)$ is called the lifted vector of $(z_1, z_2, ..., z_m) \in (S^1)^m$. Let $f$ be a circle homeomorphism with lift $F$. We define the cross-ratio of the homeomorphism $f$ of a four-tuple $(z_1, z_2, z_3, z_4)$, $z_i \in S^1$, $i = 1, ..., 4$ $z_1 \prec z_2 \prec z_4 \prec z_1$, with respect to $f$ to be $Dist(z_1, z_2, z_3, z_4; f) = Dist(\widehat{z}_1, \widehat{z}_2, \widehat{z}_3, \widehat{z}_4; F)$, where $(\widehat{z}_1, \widehat{z}_2, \widehat{z}_3, \widehat{z}_4)$ is the lifted vector of the four-tuple $(z_1, z_2, z_3, z_4)$.

# 3. Distortion lemmas and covering intervals theorem

In this section, we estimate the distortion of cross-ratios of four points. Let $\omega(\delta; f)$ denotes a modulus of continuity of $f$ in the closed interval $I$, that is $\omega(\delta; f) = \{\sup |f(x_1) - f(x_2)|$ for $x_1, x_2 \in I$, $|x_1 - x_2| \leq \delta\}$. If $f'$ satisfies (1) then $\omega(\delta; f') = o(\delta \log \frac{1}{\delta})$ (see Zygmund (2002)).

**Lemma 3.1.** *Suppose that a circle homeomorphism $f$ satisfies the hypotheses of Theorem 1.2. Suppose also that $z_i \in S^1, i = 1, ..., 4$ $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and the interval $[z_1, z_4]$ does not contain any break point of $f$. Then*

$$|Dst(z_1, z_2, z_3, z_4; f) - 1| \leq C_2|z_4 - z_1|\psi(|z_4 - z_1|) + |f'(z_4) - f'(z_1)|\omega(|z_4 - z_1|; f') \tag{2}$$

*where the constant $C_2$ depends on $f$.*

*Proof.* We note that if $f'$ satisfies (1) then for each $x, y \in S^1$

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{x - y}\int_x^y f'(t)dt = \frac{f'(x) + f'(y)}{2} + \mathcal{O}(|z_1 - z_4|\psi(|z_1 - z_4|)).$$

This equality is proven in the same way that of Melo and Strien (1993). Using this equality, we get

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1}\frac{z_4 - z_2}{f(z_4) - f(z_2)} = \frac{f'(z_2) + f'(z_1) + \mathcal{O}(|z_2 - z_1|\psi(|z_2 - z_1|))}{f'(z_4) + f'(z_2) + \mathcal{O}(|z_4 - z_2|\psi(|z_4 - z_2|))} = \tag{3}$$

$$\Big(1 - \frac{f'(z_4) - f'(z_1)}{f'(z_4) + f'(z_2)}\Big)\Big(1 + \mathcal{O}(|z_4 - z_1|\psi(|z_4 - z_1|))\Big) = \Big(1 - \frac{f'(z_4) - f'(z_1)}{2f'(z_4)}\frac{1}{1 - \frac{f'(z_4) - f'(z_2)}{2f'(z_4)}}\Big) \times$$

$$\Big(1 + \mathcal{O}(|z_4 - z_1|\psi(|z_4 - z_1|))\Big) = \Big(1 - \frac{f'(z_4) - f'(z_1)}{2f'(z_4)}(1 + \mathcal{O}(f'(z_4) - f'(z_2)))\Big) \times$$

$$\Big(1 + \mathcal{O}(|z_4 - z_1|\psi(|z_4 - z_1|))\Big) = 1 - \frac{f'(z_4) - f'(z_1)}{2f'(z_4)} + \mathcal{O}(|z_4 - z_1|\psi(|z_4 - z_1|).$$

In the same way can get that

$$\frac{z_3 - z_1}{f(z_3) - f(z_1)}\frac{f(z_4) - f(z_3)}{z_4 - z_3} = 1 + \frac{f'(z_4) - f'(z_1)}{2f'(z_4)} + \mathcal{O}(|z_4 - z_1|\psi(|z_4 - z_1|). \tag{4}$$

From (3) and (4) we obtain

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1}\frac{z_4 - z_2}{f(z_4) - f(z_2)}\frac{z_3 - z_1}{f(z_3) - f(z_1)}\frac{f(z_4) - f(z_3)}{z_4 - z_3} = 1 - \Big(\frac{f'(z_4) - f'(z_1)}{2f'(z_4)}\Big)^2$$
$$+\mathcal{O}(|z_4 - z_1|\psi(|z_4 - z_1|).$$

Hence, from this equality and the modulus of continuity of $f'$ follows that

$$|Dst(z_1, z_2, z_3, z_4; f) - 1| \leq const|z_4 - z_1|\psi(|z_4 - z_1|) + |f'(z_4) - f'(z_1)|\omega(|z_4 - z_1|; f').$$

The lemma is proved with $const = C_2$. □

Now we consider the case when the interval $[z_1, z_4]$ contains just one break point $b_{i_0}$. More precisely, suppose that $b_{i_0}$ lies outside the middle interval, that is $b_{i_0} \in [z_1, z_2] \cup [z_3, z_4]$. Suppose for definiteness that $b_{i_0} \in [z_1, z_2]$. We define the numbers $\alpha$, $\beta$, $\gamma$, $\tau$, $\eta$ and $z$ as follows: $\alpha := z_2 - z_1$, $\beta := z_3 - z_2$, $\gamma := z_4 - z_3$, $\tau := z_2 - b_{i_0}$, $\eta := \frac{\beta}{\alpha}$, $\xi := \frac{\tau}{\alpha}$.

**Lemma 3.2.** *Suppose that a circle homeomorphism $f$ satisfies the hypotheses of Theorem 1.2. Let $z_i \in S^1$, $i = 1, ..., 4$ with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. Suppose also that $b_{i_0} \in [z_1, z_2]$ and the other break points of $f$ are not contained in $[z_1, z_4]$. Then*

$$\Big| Dst(z_1, z_2, z_3, z_4; f) - \frac{(\sigma(b_{i_0}) + (1 - \sigma(b_{i_0})\xi))(1 + \eta)}{\sigma(b_{i_0}) + (1 - \sigma(b_{i_0}))\xi + \eta} \Big| \tag{5}$$

$$\leq C_3 |z_4 - z_1| \psi(|z_4 - z_1|) + \omega(|z_4 - z_1|; f')$$

*where the constant $C_3 > 0$ depends on $f$.*

*Proof.* By assumption $b_{i_0} \in [z_1, z_2]$. Rewriting $Dst(z_1, z_2, z_3, z_4; f)$ in the form

$$Dst(z_1, z_2, z_3, z_4; f) = \frac{Cr(f(z_1), f(z_2), f(z_3), f(z_4))}{Cr(z_1, z_2, z_3, z_4)} =$$

$$\Big( \frac{f(z_2) - f(z_1)}{z_2 - z_1} \cdot \frac{z_3 - z_1}{f(z_3) - f(z_1)} \Big) \Big( \frac{f(z_4) - f(z_3)}{z_4 - z_3} \cdot \frac{z_4 - z_2}{f(z_4) - f(z_2)} \Big)$$

it is easy to check, that each multiplication in brackets equals to the following

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \cdot \frac{z_3 - z_1}{f(z_3) - f(z_1)} = \frac{f'_+(b_{i_0})(z_2 - x_b) + f'_-(b_{i_0})(x_b - z_1)}{z_2 - z_1} \times \tag{6}$$

$$\frac{z_3 - z_1}{f'_+(b_{i_0})(z_3 - x_b) + f'_-(b_{i_0})(x_b - z_1)} = \frac{(\sigma(b_{i_0}) + (1 - \sigma(b_{i_0}))\xi)(1 + \eta)}{\sigma(b_{i_0}) + (1 - \sigma(b_{i_0}))\xi + \eta},$$

where $\sigma(b_{i_0}) = \frac{f'_-(b_{i_0})}{f'_+(b_{i_0})}$ the jump ratio of $f$ at the point $b_{i_0}$.

$$\frac{f(z_4) - f(z_3)}{z_4 - z_3} \cdot \frac{z_4 - z_2}{f(z_4) - f(z_2)} = \Big( \frac{f'(z_4) + f'(z_3)}{2} + \mathcal{O}(|z_4 - z_3| \psi(|z_4 - z_3|)) \Big) : \tag{7}$$

$$\Big( \frac{f'(z_4) + f'(z_2)}{2} + \mathcal{O}(|z_4 - z_2| \psi(|z_4 - z_2|)) \Big) = \frac{f'(z_4) + f'(z_3) + \mathcal{O}(|z_4 - z_3| \psi(|z_4 - z_3|))}{f'(z_4) + f'(z_2) + \mathcal{O}(|z_4 - z_2| \psi(|z_4 - z_2|))} =$$

$$1+\frac{f'(z_3)-f'(z_2)}{f'(z_4)+f'(z_2)}+\mathcal{O}(|z_4-z_1|\psi(|z_4-z_1|))$$

From (6) and (7) we have

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}\frac{z_3-z_1}{f(z_3)-f(z_1)}\frac{f(z_4)-f(z_3)}{z_4-z_3}\frac{z_4-z_2}{f(z_4)-f(z_2)}=\frac{(\sigma(b_{i_0})+(1-\sigma(b_{i_0}))\xi)(1+\eta)}{\sigma(b_{i_0})+(1-\sigma(b_{i_0}))\xi+\eta}+$$

$$\frac{f'(z_3)-f'(z_2)}{f'(z_4)+f'(z_2)}\frac{(\sigma(b_{i_0})+(1-\sigma(b_{i_0}))\xi)(1+\eta)}{\sigma(b_{i_0})+(1-\sigma(b_{i_0}))\xi+\eta}+\mathcal{O}(|z_4-z_1|\psi(|z_4-z_1|)).$$

Hence, from this equality and the modulus of continuity of $f'$ follows that

$$\Big|Dst(z_1,z_2,z_3,z_4;f)-\frac{(\sigma(b_{i_0})+(1-\sigma(b_{i_0})\xi))(1+\eta)}{\sigma(b_{i_0})+(1-\sigma(b_{i_0}))\xi+\eta}\Big|$$

$$\leq const|z_4-z_1|\psi(|z_4-z_1|)+\omega(|z_4-z_1|;f').$$

The lemma is proved with $const=C_3$. □

# 4. Covering intervals theorem

Now we provide a theorem on covering intervals for circle homeomorphisms with break points. Consider $f\in S^1\setminus\{b(1),b(2),...,b(n)\}$ with $n$ break points and irrational rotation number $\rho$. Suppose that all these break points lie in different orbits. If this were not the case, then we could achieve it by considering sufficiently high renormalizations. We set $B(f)=\{b(1),b(2),...,b(n)\}$.

We introduce the notion of a 'regular' cover of the break points in $B(f)$. Suppose that $z_i\in S^1, i=1,...,4,\ z_1\prec z_2\prec z_3\prec z_4\prec z_1$ and $r_n$ takes value in the set $\{q_{n-1},q_n,q_{n-1}+q_n\}$. Suppose that the interval $[z_1,z_4]$ is $r_n$-small that is, the intervals $f^j((z_1,z_4))$, $0\leq j<r_n$ are pairwise disjoint. Suppose that the system of intervals $\{f^j([z_1,z_4]),0\leq j<r_n\}$ cover the elements of some non-empty subset $\widehat{B}\subset B(f)$. We denote the number of elements of $\widehat{B}$ by $m$. For every element $b(i_s)\in\widehat{B}$ there exists a number $l_{i_s}$, $0\leq l_{i_s}<r_n$ such that $\overline{b}^{(n)}(i_s)=f^{-l_{i_s}}(b(i_s))\in[z_1,z_4]$. the point $\overline{b}^{(n)}(i_s)$ is called the $r_n-pre-image$ of the element $b(i_s)$ in $[z_1,z_4]$. The set of $r_n$ pre-image of elements of $\widehat{B}$ also consists of $m$ elements: $\overline{b}^{(n)}(i_1),\overline{b}^{(n)}(i_2),...,\overline{b}^{(n)}(i_m)$; we denote the maximal element of this set by $\widehat{b}_t^{(n)}$. Clearly, $\widehat{b}_t^{(n)}=\widehat{b}^{(n)}(i_t)$ for some $0\leq t\leq m$. We introduce the following notations:

$$\eta(j)=\frac{l([f^j(z_2),f^j(z_3)])}{l([f^j(z_1),f^j(z_2)])},\quad z^{(i_s)}(j)=\frac{l([f^j(\widehat{b}^{(n)}(i_s)),f^j(z_2)])}{l([f^j(z_1),f^j(z_2)])}$$

$1 \leq s \leq m$, $0 \leq j < r_n$.

In cases where $\widehat{b}^{(n)}(i_s) \in [z_1, z_2]$, the numbers $z^{i_s}(j)$ are called normalized coordinates of the elements $f^j(\widehat{b}^{(n)}(i_s))$. When the point $\widehat{b}^{(n)}(i_s)$ moves from $z_2$ to $z_1$, the normalized coordinate $z^{i_s}(j)$ varies from 0 to 1. It is easy to see that

$$e^{-v}\eta(0) \leq \eta(j) \leq e^{v}\eta(0), \quad e^{-v}z^{i_s}(0) \leq z^{i_s}(j) \leq e^{v}z^{i_s}(0),\ i = 1, 2, ..., n$$

for all $1 \leq j < r_n$.

**Definition 4.1.** *Let $K > M \geq 1$, $\zeta \in (0, 1)$, $\delta > 0$ be constant numbers, let $n$ be a positive integer and let $x_0 \in S^1$. We say a triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$, $z_i \in S^1$, $i = 1, ...4$ $(K, M, \delta, \zeta, x_0)-$regularly cover the break points in a subset $\widehat{B}$ if for some $r_n \in \{q_{n-1}, q_n, q_{n-1} + q_n\}$ the following conditions hold:*

**1)** $[z_1, z_4] \subset (x_0 - \delta, x_0 + \delta)$ *and the system of intervals* $\{f^j([z_1, z_4]), 0 \leq j \leq r_n - 1\}$ *covers every point in* $\widehat{B}$ *only once.*

**2)** $z_2 = \widehat{b}_t^{(n)}$ *and* $\bar{b}^{(n)}(i_s) \in [z_1, z_2),\ 1 \leq s \leq n, s \neq t.$

**3)** $Ml([z_2, z_3]) \leq l([z_1, z_2]) \leq Kl([z_2, z_3])$ *and* $K^{-1}l([z_3, z_4]) \leq l([z_2, z_3]) \leq Kl([z_3, z_4])$.

**4)** *The lengths of the intervals* $f^{r_n}([z_1, z_2])$, $f^{r_n}([z_2, z_3])$ *and* $f^{r_n}([z_3, z_4])$ *are pairwise $K-$commensurable.*

**5)** $\max\{l([f^{r_n}(z_i), x_0]), l([z_i, x_0]), i = 1, ..., 4\} \leq Kl([z_1, z_2])$.

**6)** $\max\limits_{1 \leq s \leq m}\{z^{(i_s)}(0)\} < \zeta$.

**Definition 4.2.** *A subset $\widehat{B} \subset \{b(1), b(2), ..., b(n)\}$ is said to be non-trivial if* $\prod\limits_{b \in \widehat{B}} \sigma(b, F) \neq 1$.

We now state a theorem on covering intervals which plays key role in the proof of main result. The proof of this theorem does not depend on the considered class of circle homeomorphisms and similar to the proof of Dzhalilov et al. (2012). That is why here we provide this theorem without proof.

**Theorem 4.1.** *Suppose that a homeomorphism $f$ satisfies the hypotheses of Theorem 1.2. Let $x_0 \in S^1$ and let $M \geq 1$, $\delta, \zeta \in (0, 1)$ be constant numbers. Then there exists a constant $K = K(f, M, \zeta) > M$ such that for any sufficiently* *large $k$ there exists non-trivial subset $\widehat{B}=\widehat{B}(k)=\{b(i_1),b(i_2),...,b(i_m)\}$, points $z_i\in S^1,\ i=1,...,4,\ z_1\prec z_2\prec z_3\prec z_4\prec z_1$ and $r_n=r_n(z_1,z_2,z_3,z_4)\in\{q_{n-1},q_n,q_{n-1}+q_n\}$ such that the intervals $[z_s,z_{s+1}],\ s=1,2,3\ (K,M,\delta,\zeta,x_0)-$regularly cover the break points of $\widehat{B}$.*

## 5. Proof of main result

We state three lemmas and use them to prove the main result. These lemmas can be proved in similar way that of Dzhalilov et al. (2012).

**Lemma 5.1.** *Suppose that at a point $x=x_0$ the lift $\Psi$ of the conjugation $\varphi$ has positive derivative, $\Psi'(x_0)=\omega_0$. Then there exist $\delta=\delta(x_0,\epsilon)>0$ for any $\epsilon>0$ and a constant $C_3=C_3(R_1,\omega_0)$ for any $R_1>1$ such that the inequality*

$$|Dist(z_1,z_2,z_3,z_4;\Psi)-1|<C_3\epsilon$$

*holds for any points $z_1<z_2<z_3<z_4$ lying in the neighborhood $(x_0-\delta,x_0+\delta)$ and satisfying the conditions:*

**(i)** *the intervals $[z_1,z_2],[z_2,z_3],[z_3,z_4]$ are pairwise $R_1-$commensurable.*

**(ii)** $\max\{|z_1-x_0|,|z_4-x_0|\}\le R_1|z_1-z_2|$.

We define the following functions on the domain $\{(x,y):x>0,\ 0\le y\le 1\}$ $F_i(x,y)=\frac{[\sigma_i+(1-\sigma_i)y](1+x)}{\sigma_i+(1-\sigma_i)y+x},\quad i=1,2,...,n$ where the $\sigma_i$ are the jumps of $f$ at the points $b(i)$.

**Lemma 5.2.** *Let $\{b(i_1),b(i_2),...,b(i_m)\}$ be an arbitrary non-trivial subset of break points of $f$, so that $\prod_{s=1}^{m}\sigma_{i_s}=A\neq 1$. Then there exist constants $\Omega_0=\Omega_0(\sigma_{i_1},\sigma_{i_2},...,\sigma_{i_m})>1$ and $\tau_0=\tau_0(\sigma_{i_1},\sigma_{i_2},...,\sigma_{i_m})\in(0,1)$ such that the inequality*

$$\left|\prod_{s=1}^{m}F_{i_s}(x_s,y_s)-A\right|\le\frac{|A-1|}{4}$$

*holds for all $x_s\ge\Omega_0,\ y_s\in[0,\tau_0],\ s=1,2,...,m$.*

We use $\tau_0$ and $\Omega_0$ to define two new constants $\overline{\tau_0}$ and $\overline{\Omega_0}$, which will play an important role in the proof of Theorem 1.2. We set $\overline{\tau_0}=\min\tau_0(\sigma_{i_1},\sigma_{i_2},...,\sigma_{i_m})\in(0,1)$, $\overline{\Omega_0}=\max\Omega_0(\sigma_{i_1},\sigma_{i_2},...,\sigma_{i_m})$ where the minimum and maximum are taken over all non-trivial subsets $\{b(i_1),b(i_2),...,b(i_m)\}$ of break points of $f$.

*Proof of Theorem 1.2.* Suppose that a homeomorphism $f$ satisfies the hypotheses of the theorem. Since the rotation number $\rho$ is irrational, the invariant measure $\mu$ has no atoms and the conjugation $\varphi(x)$ is given by monotonic function $\mu([0,x]),\ x \in S^1$. The finite derivative $\Psi'(x)$ of the lift exists by the monotonicity of the function $\Psi$ for almost all $x$ with respect to Lebesgue measure. We claim that $\Psi'(x) = 0$ at all points $x$ where the finite derivative exists. Suppose that $\Psi'(x_0) = \omega_0 > 0$ at some point $x_0 \in S^1$. We fix $\epsilon > 0$. Let $\delta = \delta(x_0, \epsilon) > 0$ be defined by Lemma 5.1. We use the constants $\overline{\Omega_0}$ and $\overline{\tau_0}$ to define new constants: $M_0 = \overline{\Omega_0}e^v$, $\zeta_0 = \overline{\tau_0}e^v$ where $v > 0$ is the total variation of $\ln F'$ over $S^1$. Let $K_0 = K_0(f, M_0, \zeta_0) > M_0 > 1$ be the constant defined in the assertion of Theorem 3.1. By that assertion, for sufficiently large $n$ there exist a non-trivial subset $\widehat{B} = \{b(i_1), b(i_2), ..., b(i_m)\}$ of singular points of $f$, points $z_i \in S^1$, $i = 1, ..., 4$ $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and a number $r_n \in \{q_{n-1}, q_n, q_{n-1}+q_n\}$ such that the triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$ $(K_0, M_0, \delta, \zeta_0, x_0)-$regularly cover the points in $\widehat{B}$.

Since after $r_n$ steps the images of the triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$ cover all points of the non-trivial subset $\widehat{B}$, the cross-ratio $Cr(z_1, z_2, z_3, z_4)$ and $Cr(f^{r_n}(z_1), f^{r_n}(z_2), f^{r_n}(z_3), f^{r_n}(z_4))$ are substantially different. More precisely, the following lemma holds.

**Lemma 5.3.** *The inequality*

$$|Dist(z_1, z_2, z_3, z_4; f^{r_n}) - 1| \geq R_2 \tag{8}$$

*holds for sufficiently large $n$, where the constant $R_2 > 0$ depends only on $f$.*

Since the intervals $[z_s, z_{s+1}],\quad s = 1, 2, 3$ $(K_0, M_0, \delta, \zeta_0, x_0)-$regularly cover the points in $\widehat{B}$ these intervals along with $[f^{r_n}(z_s), f^{r_n}(z_{s+1})],\quad s = 1, 2, 3$ satisfy conditions $(i)$, $(ii)$ of Lemma 4.1 with constant $R_1 = K_0$. Using the assertion of Lemma 4.1 we obtain

$$|Dist(z_1, z_2, z_3, z_4; \varphi) - 1| \leq C_4\epsilon \tag{9}$$

$$|Dist(f^{r_n}(z_1), f^{r_n}(z_2), f^{r_n}(z_3), f^{r_n}(z_4); \varphi) - 1| \leq C_4\epsilon \tag{10}$$

where the constant $C_4 > 0$ depends on $R_1$ and $\omega$.

Since $\varphi$ effects a conjugation to a linear rotation, it is easy to see that

$$Cr(\varphi(f^{r_n}(z_1)), \varphi(f^{r_n}(z_2)), \varphi(f^{r_n}(z_3)), \varphi(f^{r_n}(z_4))) = Cr(\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)) \tag{11}$$

Formulae (9) − (11) immediately imply that

$$|Dist(z_1, z_2, z_3, z_4; f^{r_n}) - 1| \leq C_5 \epsilon \tag{12}$$

where the constant $C_5 > 0$ is independent of $\epsilon$ and $n$. The relations (12) and (8) cannot hold simultaneously for sufficiently small $\epsilon$. This contradiction proves Theorem 1.2.

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